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Convective heat transport in a fluid layer of infinite Prandtl number: upper bounds for the case of rigid lower boundary and stress-free upper boundary

N.K. Vitanov[a]

Max-Planck-Institut für Physik Komplexer Systeme, Nöthnitzer Strasse 38, 01187 Dresden, Germany

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Abstract. We present the theory of the multi-α-solutions of the variational problem for the upper bounds on the convective heat transport in a heated from below horizontal fluid layer with rigid lower boundary and stress-free upper boundary. A sequence of upper bounds on the convective heat transport is obtained. The highest bound Nu $= 1 + (1/6)R^{1/3}$ is between the bounds Nu $= 1 + 0.152R^{1/3}$ for the case of a fluid layer with two rigid boundaries and Nu $= 1 + 0.3254R^{1/3}$ for the case of a fluid layer with two stress-free boundaries. As an additional result of the presented theory we obtain small corrections of the boundary layer thicknesses of the optimum fields for the case of fluid layer with two rigid boundaries. These corrections lead to systematically lower upper bounds on the convective heat transport in comparison to the bounds obtained in [5].

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1 Introduction

The optimum theory of the turbulence is one of the few tools for obtaining rigorous estimates of the turbulent quantities directly from the Navier-Stokes equations. We do not know turbulent solutions of the Navier-Stokes equations and full numerical simulations of the turbulent flows with very large Reynolds or Rayleigh numbers are out of reach today. Thus the interest and research activities in the area of the optimum theory of turbulence show a steady increasing in the last decade. The first upper bounds on the turbulent transport quantities were obtained in 1963 when the ideas of Malkus [1,2] stimulated Howard [3] to formulate the theory of the upper bounds on the heat transport in a layer of fluid, heated from below. He used a solution of the corresponding variational problem with a single wave number. Busse [4] introduced the multi-α-solutions of the variational problem. The Howard-Busse method has been further developed by Chan [5] who considered intermediate sublayers between the boundary and internal sublayers of the fields connected to the wave numbers of the multi-α-solution of the variational problem for the upper bounds on the convective heat transport in a fluid layer of infinite Prandtl number. Many results have been obtained by the Howard-Busse method [6–14]. and some of them have stimulated the rapid development of the optimum theory of turbulence in several directions. The most important direction is formulation of new methods and amendments of the existing ones. In 1992 Doering and Constantin developed the second method of the optimum theory of turbulence [15]. Its main idea is to decompose the velocity into a steady background field which carries the inhomogeneous boundary conditions and a homogeneous fluctuations field. The method leads to quick estimations of the turbulent quantities and has been applied to convection and shear flows [16–20]. An energy-balance parameter modification of the Doering-Constantin method was proposed by Nicodemus, Grossmann and Holthaus [21]. Up to now this modification has been applied for obtaining upper bounds on the energy dissipation in turbulent shear flow and in Couette-Ekman flow [22–25]. Another direction of the development of the theory is connected to clarifying the relations between the methods and formulations of the variational problems for the Navier-Stokes equations [26–29]. Finally the optimum theory of turbulence finds applications for new systems and new boundary conditions. For an example the theory has been applied in the plasma physics where upper bounds on the heat transport due to ion-temperature gradient, on the energy dissipation in a turbulent pinch, etc have been calculated [30–36].

In this paper we obtain upper bounds on the convective heat transport in a heated from below horizontally infinite fluid layer. Chan [5] developed the theory for the case of fluid layer with two rigid boundaries. His result is that the convective heat transport is bounded by $0.152R^{1/3}$ as the number of the wave numbers of the multi-α-solution and the Rayleigh number R tend to infinity together. In this paper we discuss the case of fluid layer with rigid lower

[a] e-mail: vitanov@mpipks-dresden.mpg.de

boundary and stress-free upper boundary and obtain a sequence of bounds the highest of which is $(1/6)R^{1/3}$.

The paper is organised as follows. In Section 2 we formulate briefly the variational problem. In Section 3 we obtain upper bounds on the convective heat transport on the basis of the multi-α-solutions of the variational problem as well as relationships for the wave numbers and boundary layer thicknesses of the optimum fields. In the last section we discuss the obtained results and show how on the their basis the bound obtained by Chan [5] for the rigid-rigid case can be slightly but systematically lowered. In the appendix we show that the onset of the thermal convection in a fluid layer of infinite Prandtl number is contained as solution of the Euler equations of the variational functional for the upper bounds on the convective heat transport.

2 Mathematical formulation of the problem

We consider a horizontally infinite fluid layer heated from below of thickness d with fixed temperatures T_1 and T_2 at the upper and lower boundaries and denote the coefficient of thermal expansion by γ, the kinematic viscosity by ν, the acceleration of gravity by g and the thermal diffusivity of the fluid as κ. The characteristic parameters of the system are the Rayleigh number: $R = \gamma(T_2 - T_1)gd^3/(\kappa\nu)$ and the Prandtl number: $P = \nu/\kappa$. Introducing a Cartesian system of coordinates with z-axis in the vertical direction and using d as length scale, d^2/κ as time scale and $(T_2 - T_1)/R$ as temperature scale we can write the Navier-Stokes equations for the velocity vector $\mathbf{u}$ and the heat equation for the deviation Θ from the static temperature distribution in dimensionless form,

$$(1/P)\left(\partial\mathbf{u}/\partial t + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mathbf{k}\Theta + \nabla^2\mathbf{u} \quad (1)$$

$$\nabla\cdot\mathbf{u} = 0 \quad (2)$$

$$\partial\Theta/\partial t + \mathbf{u}\cdot\nabla\Theta = R\mathbf{k}\cdot\mathbf{u} + \nabla^2\Theta \quad (3)$$

$\mathbf{k}$ is the vertical unit vector. Denoting the z-component of $\mathbf{u}$ as w we write for the rigid boundary conditions on the bottom of the layer: $w = \partial w/\partial z = \Theta = 0$ at $z = -1/2$ and for the stress-free boundary conditions on the top of the layer: $w = \partial^2 w/\partial z^2 = \Theta = 0$ at $z = 1/2$.

We shall use the averages over the the planes $z = \text{const}$ (denoted as $\overline{q}$) and over the fluid layer (denoted as $\langle q\rangle$). Denoting the horizontal size of the fluid layer as L and the limes when $L \to \infty$ as lim we define

$$\overline{q} = \lim(1/(4L^2))\int_{-L}^{L}\int_{-L}^{L}\mathrm{d}x\mathrm{d}y\; q(x,y,z,t) \quad (4)$$

$$\langle q\rangle = \lim(1/(4L^2))\int_{-L}^{L}\int_{-L}^{L}\int_{-1/2}^{1/2}\mathrm{d}x\mathrm{d}y\mathrm{d}z\; q(x,y,z,t). \quad (5)$$

The temperature field is separated into two parts $\Theta = \overline{\Theta} + T$ such that $\overline{T} = 0$ holds. We average over the fluid layer the two equations obtained by a multiplication of (1) by $\mathbf{u}$ and by a subtraction of the horizontal average of (3) from (3). Thus we obtain the relationships

$$(1/2P)\mathrm{d}\langle\mathbf{u}\cdot\mathbf{u}\rangle/\mathrm{d}t = \langle wT\rangle - \langle|\nabla\mathbf{u}|^2\rangle \quad (6)$$

$$(1/2)\mathrm{d}\langle T^2\rangle/\mathrm{d}t = R\langle wT\rangle - \langle|\nabla T|^2\rangle - \langle\overline{wT}\partial\overline{\Theta}/\partial z\rangle. \quad (7)$$

We are interested in turbulent convection long after any external parameter has been changed. We define this situation by the condition that all horizontally averaged quantities are time independent. In this case we obtain from the first integral of the horizontal averaged (3): $\mathrm{d}\overline{\Theta}/\mathrm{d}z = \overline{wT} - \langle wT\rangle$. Using this relationship we obtain the power integrals

$$\langle wT\rangle = \langle|\nabla\mathbf{u}|^2\rangle \quad (8)$$

$$\langle|\nabla T|^2\rangle = R\langle wT\rangle + \langle wT\rangle^2 - \langle\overline{wT}^2\rangle \quad (9)$$

which hold for all Prandtl numbers P. The imposition of the infinite Prandtl number condition restricts further the fields that satisfy the power integrals. (1) becomes linear in the limit $P \to \infty$ and we incorporate it as an additional constraint into the variational problem. The pressure is eliminated by taking the z-component of the double curl of (1). Thus we obtain the relationship: $\nabla_1^2 T + \nabla^4 w = 0$. We take the equation of continuity as a constraint into the variational problem by means of the general representation of a solenoidal vector field $\mathbf{u}$ in terms of a poloidal and a toroidal component $\mathbf{u} = \nabla\times(\nabla\times\mathbf{k}\phi) + \nabla\times\mathbf{k}\psi$ where the condition $\overline{\phi} = \overline{\psi} = 0$ can be imposed without changing $\mathbf{u}$. Taking the curl of (1) we see that ψ must vanish in the limit of infinite Prandtl number. The z-component of $\mathbf{u}$ is given by the poloidal field ϕ, $w = -\nabla_1^2\phi$ where $\nabla_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

We write a functional for the convective heat transport $\mathrm{Nu} - 1 = \langle wT\rangle/R$ where Nu is the Nusselt number. Using (8, 9) and imposing the normalisation condition $\langle w\theta\rangle = 1$ we obtain the following variational problem:

Given Rayleigh number R find the maximum $F(R)$ of the variational functional

$$\mathcal{F}(R,w,\theta) = [1 - (1/R)\langle|\nabla\theta|^2\rangle]/\langle(1 - \overline{w\theta})^2\rangle \quad (10)$$

among all fields w and θ subject to the constraints $\langle w\theta\rangle = 1$ and $\nabla^4 w + \nabla_1^2\theta = 0$ and the boundary conditions $w = \partial w/\partial z = \theta = 0$ at $z = -1/2$ and $w = \partial^2 w/\partial z^2 = \theta = 0$ at $z = 1/2$.

The Euler equations for the functional (10) are

$$(1/RF)\nabla^6\theta + \nabla^4\left[\left(1 - \overline{w\theta} - 2\lambda/F\right)w\right] + (1 - \overline{w\theta})\nabla^4 w = 0 \quad (11)$$

$$\nabla^4 w + \nabla_1^2\theta = 0 \quad (12)$$

$$-1 \le \lambda = -1/2\left(2 - (1/R)\langle|\theta|^2\rangle\right) \le -1/2. \quad (13)$$

We note that the Euler equations contain the solution that describes the onset of the convection (see Appendix A).

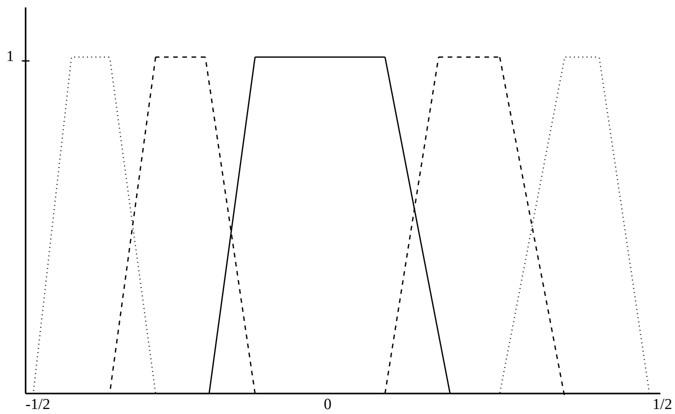

Fig. 1. Assumed layer structure for the case of a multi-α-solution. Rigid line: schematic representation of $w_1\theta_1$. Dashed line: schematic representation of $w_2\theta_2$. Dotted line: schematic representation of $w_3\theta_3$.

3 The multi-α-solutions

When R is large enough $2\lambda/F \to 0$ and $\overline{w\theta} \approx 1$ almost in the whole fluid layer. We assume also $\nabla^6\theta \ll RF$. Thus (11) is approximately satisfied. In order to solve (12) we write w and θ as multi-α-solutions of Busse [4]

$$w = \sum_{n=1}^{N} w_n(z)\phi_n(x,y) \quad \theta = \sum_{n=1}^{N} \theta_n(z)\phi_n(x,y) \tag{14}$$

where $\nabla_1\phi_n = -\alpha_n^2\phi_n$ and $\phi_n\phi_m = \delta_{nm}$. $\nabla_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and δ_{nm} is the Kronecker delta symbol. Thus we obtain the system of equations

$$(\mathrm{d}^2/\mathrm{d}z^2 - \alpha_n^2)^2 w_n - \alpha_n^2\theta_n = 0. \tag{15}$$

With respect to the midplane of the fluid layer we assume two systems of layers: layers from the midplane to the lower rigid boundary of the fluid layer and layers from the midplane to the upper stress-free boundary of the fluid layer (see Fig. 1). For each of the two systems of layers we have the sublayers structure as follows. Each of the components of $\overline{w\theta}$ connected to the different wave numbers α_n is significantly different from 0 in a confined region of the fluid layer. This region can be divided into three subregions. Two of these three subregions are the areas of coexistence of the neighbouring modes. We assume that $\overline{w_i\theta_i}$ and $\overline{w_{i-1}\theta_{i-1}}$ corresponding to the ith mode and $i-1$th mode coexist only in the $(i-1)$-boundary layer *i.e.* the $(i-1)$-boundary layer coincides with the ith internal layer. Thus we have two sets of intermediate and boundary layers. For the lower set (from the midplane of the fluid layer to the rigid lower boundary) we introduce the coordinates $\xi_n^{\mathrm{l}} = \alpha_n(z-1/2)$ for the intermediate layers and $\eta_n^{\mathrm{l}} = (\alpha_n/\delta_n^{\mathrm{l}})(1/2-z)$ for the boundary layers. For the upper set of layers the coordinates are $\xi_n^{\mathrm{u}} = \alpha_n(1/2-z)$ for the intermediate layers and $\eta_n^{\mathrm{u}} = (\alpha_n/\delta_n^{\mathrm{u}})(1/2-z)$ for the boundary layers. We define $\alpha_0 = \delta_0^{\mathrm{u,l}} = 1$. Note that the quantities $\delta_n^{\mathrm{u,l}}$ can be considered as characteristic lengths for the corresponding boundary layers. The internal layer of the field $\overline{w_n\theta_n}$ coincides with the boundary layer of the field $\overline{w_{n-1}\theta_{n-1}}$. The two fields coexist in such a way that $\overline{w\theta} \approx 1$ when $R \to \infty$ *i.e.* we have the relationships

$$1 - \hat{w}_{n-1}(\eta_{n-1}^{\mathrm{u,l}})\hat{\theta}_{n-1}(\eta_{n-1}^{\mathrm{u,l}}) - \tilde{w}_n(\eta_{n-1}^{\mathrm{u,l}})\tilde{\theta}_n(\eta_{n-1}^{\mathrm{u,l}}) = O(R^{-1}). \tag{16}$$

Thus we ensure that $1-\overline{w\theta}$ vanishes through the fluid layer except in the two boundary layers near the fluid boundaries and the denominator $\langle(1-\overline{w\theta})^2\rangle$ of the variational functional (10) is made as small as possible.

For the layers with coordinates $\eta_{n-1}^{\mathrm{u,l}}$ we assume that the terms containing the derivatives $\mathrm{d}^2/\mathrm{d}{\eta_{n-1}^{\mathrm{u,l}}}^2$ are much smaller than the terms containing the wave numbers α_n. Thus we obtain the solutions: $w_n(z) = C_n\tilde{w}_n(\eta_{n-1}^{\mathrm{u,l}})$; $\theta_n(z) = (1/C_n)\tilde{\theta}_n(\eta_{n-1}^{\mathrm{u,l}})$ with $C_n = 1/\alpha_n$. In the layers with coordinates $\xi_n^{\mathrm{u,l}}$ we search for solutions of the form: $w_n(z) = B_n\breve{w}_n(\xi_n^{\mathrm{u,l}})$; $\theta_n(z) = (1/B_n)\breve{\theta}_n(\xi_n^{\mathrm{u,l}})$ with $B_n = 1/\alpha_n$. Using that in these layers $w_n\theta_n \approx 1$ *i.e.* $\theta_n = 1/w_n$ we have to solve the equations

$$\breve{w}_n(\mathrm{d}^2/\mathrm{d}{\xi_n^{\mathrm{u,l}}}^2 - 1)^2\breve{w}_n = 1. \tag{17}$$

For the upper layers we have the boundary conditions: $\breve{w}_n(0) = \breve{w}_n''(0) = 0$ plus the matching conditions that $\breve{w}_n(\xi_n^{\mathrm{u}} \to \infty)$ must match $\tilde{w}_n(\eta_{n-1}^{\mathrm{u}} \to 0)$. The solution is: $\breve{w}_n = c\xi_n^{\mathrm{u}} - ({\xi_n^{\mathrm{u}}}^3/(6c))\ln(1/\xi_n^{\mathrm{u}})$ with $c = 0.83421$. For the lower layers the boundary conditions are: $\breve{w}_n(0) = \breve{w}_n'(0) = 0$ and $\breve{w}_n(\xi_n^{\mathrm{l}} \to \infty)$ must match $\tilde{w}_n(\eta_{n-1}^{\mathrm{l}} \to 0)$. The solution is: $\breve{w}_n = {\xi_n^{\mathrm{l}}}^2\sqrt{\ln(1/{\xi_n^{\mathrm{l}}}^2)}$.

For the upper and lower boundary layers we search the solutions in the form $w_n = A_n^{\mathrm{u,l}}\hat{w}_n(\eta_n^{\mathrm{u,l}})$; $\theta_n = (1/A_n^{\mathrm{u,l}})\hat{\theta}_n(\eta_n^{\mathrm{u,l}})$. In these layers the field profiles increase sharply from 0 to their value at the bounds between the corresponding boundary and intermediate layers. We assume that in the governing equations the terms containing the highest derivatives dominate over the other terms. Thus we obtain the simplified equations

$$(\alpha_n^4/{\delta_n^{\mathrm{u,l}}}^4){A_n^{\mathrm{u,l}}}^2(\mathrm{d}^4\hat{w}_n/\mathrm{d}{\eta_n^{\mathrm{u,l}}}^4) = \alpha_n^2\hat{\theta}_n. \tag{18}$$

Matching the solutions between the corresponding boundary and intermediate layers we obtain

$$A_n^{\mathrm{l}} = (1/\alpha_n){\delta_n^{\mathrm{l}}}^2\sqrt{\ln(1/\delta_n^{\mathrm{l}})}; \quad A_n^{\mathrm{u}} = c\delta_n^{\mathrm{u}}/\alpha_n. \tag{19}$$

We introduce the small parameters $\epsilon_n^{\mathrm{u}} = (\delta_n^{\mathrm{u}}/c)^2$ and $\epsilon_n^{\mathrm{l}} = 1/(\ln(1/\delta_n^{\mathrm{l}}))$ and expand the solutions as power series of $\epsilon_n^{\mathrm{u,l}}$

$$\hat{w}_n = \hat{w}_{n,0}^{\mathrm{u,l}} + \epsilon_n^{\mathrm{u,l}}\hat{w}_{n,1}^{\mathrm{u,l}} + {\epsilon_n^{\mathrm{u,l}}}^2\hat{w}_{n,2}^{\mathrm{u,l}} + ... \tag{20}$$

$$\hat{\theta}_n = \epsilon_n^{\mathrm{u,l}}\hat{\theta}_{n,0}^{\mathrm{u,l}} + {\epsilon_n^{\mathrm{u,l}}}^2\hat{\theta}_{n,1}^{\mathrm{u,l}} + {\epsilon_n^{\mathrm{u,l}}}^3\hat{\theta}_{n,2}^{\mathrm{u,l}} + ... \tag{21}$$

Thus we obtain the approximate solutions

$$\hat{w}_{n,0}^{\mathrm{l}} = {\eta_n^{\mathrm{l}}}^2; \quad \hat{w}_{n,0}^{\mathrm{u}} = \eta_n^{\mathrm{u}}. \tag{22}$$

Defining $\tilde{\theta}_{N+1} = 0$; $\langle \tilde{\theta}_1^2 \rangle_l = \langle \tilde{\theta}_1^2 \rangle_u = 1/2$ we obtain for the functional $F = T/Z$

$$T = 1 - \frac{1}{R} \sum_{n=1}^{N} \left\{ \frac{\alpha_n^3}{{\delta_n^{\rm l}}^5 \ln(1/\delta_n^{\rm l})} \int_0^\infty {\rm d}\eta_n^{\rm l} \left(\frac{{\rm d}\hat{\theta}_{n,0}}{{\rm d}\eta_n^{\rm l}} \right)^2 + \frac{\alpha_n^3}{c^2 {\delta_n^{\rm u}}^3} \int_0^\infty {\rm d}\eta_n^{\rm u} \left(\frac{{\rm d}\hat{\theta}_{n,0}}{{\rm d}\eta_n^{\rm u}} \right)^2 + \frac{\alpha_n^4 \delta_{n-1}^{\rm l}}{\alpha_{n-1}} \int_0^\infty {\rm d}\eta_{n-1}^{\rm l} \tilde{\theta}_n^2(\eta_{n-1}^{\rm l}) + \frac{\alpha^4 \delta_{n-1}^{\rm u}}{\alpha_{n-1}} \int_0^\infty {\rm d}\eta_{n-1}^{\rm u} \tilde{\theta}_n^2(\eta_{n-1}^{\rm u}) \right\} \quad (23)$$

$$Z = \sum_{n=1}^{N} \left\{ \frac{\delta_N^{\rm l}}{\alpha_N} \left[\frac{\alpha_N}{\delta_N^{\rm l}} \frac{\delta_n^{\rm l}}{\alpha_n} \int_0^\infty {\rm d}\eta_n^l (1 - {\eta_n^{\rm l}}^2)^2 \hat{\theta}_{n,0} - \tilde{\theta}_{n+1}^2)^2 \right] + \frac{\delta_N^{\rm u}}{\alpha_N} \left[\frac{\alpha_N}{\delta_N^{\rm u}} \frac{\delta_n^{\rm u}}{\alpha_n} \int_0^\infty {\rm d}\eta_n^{\rm u} (1 - \eta_n^{\rm u} \hat{\theta}_{n,0} - \tilde{\theta}_{n+1}^2)^2 \right] \right\}. \quad (24)$$

The theory of the 1-α-solution of the variational problem has been presented in [37]. Using the obtained results for the wave number, boundary layer thicknesses and upper bound on the convective heat transport connected to the optimum fields

$$\alpha_1 = {A^*}^5 (R/13)^{1/4}$$
$$A^* = [1/(1 + 0.51716 R^{-1/40} (\ln R)^{3/20})]^{1/20} \quad (25)$$
$$F_1 = 0.3404 {A^*}^{26} R^{3/10} (\ln R)^{1/5} \quad (26)$$
$$\delta_{1\rm l} = 0.38394 {A^*}^{-1} R^{-1/20} (\ln R)^{-1/5}$$
$$\delta_{1\rm u} = 1.0478 \delta_{1\rm l} (R - 13\alpha^4)/(12\alpha^4) \quad (27)$$

we can construct the quantities

$$Q_1 = \alpha_1^3 / ({\delta_1^{\rm l}}^5 \ln(1/\delta_1^{\rm l})), \quad Q_2 = \alpha_1^3 / (c {\delta_1^{\rm u}}^3). \quad (28)$$

Substituting the expressions for α_1,$\delta_1^{\rm u,l}$ we obtain that when the Rayleigh number is large enough $Q_1 \gg Q_2$ as well as $\delta_{1\rm l} \gg \delta_{1\rm u}$. Thus as a first approximation we can assume that the terms connected to the upper boundary layers except for the term connected with the 0th boundary layer are much smaller than the terms connected with the correspondent lower boundary layers. We assume that

$$\alpha_n^3 / ({\delta_n^{\rm l}}^5 \ln(1/\delta_n^{\rm l})) \propto R; \quad \alpha_n^4 \delta_{n-1}^{\rm l} / \alpha_{n-1} \propto R. \quad (29)$$

Solving (29) we obtain

$$\alpha_n = b_n \prod_{k=1}^{n-1} (\ln(1/\delta_k^{\rm l}))^{10^k / 2 \times 10^n} R^{(1/6)(2 - 5/10^n)} \quad (30)$$

$${\delta_n^{\rm l}}^{2 \times 10^n} (\ln(1/\delta_n^{\rm l}))^{4 \times 10^{n-1}} \prod_{k=1}^{n-1} (\ln(1/\delta_k^{\rm l}))^{-6 \times 10^{k-1}} = R^{-1} \quad (31)$$

where we have introduced the set of parameters b_n in order to convert the proportionalities (29) to equalities. Using (30, 31) we can write the functional F in the form

$$F_N = K_N \frac{\alpha_N}{\delta_N^{\rm l}} b_N^{-1} \quad (32)$$

where $K_N = T_1/Z_1$ and

$$T_1 = 1 - \sum_{n=1}^{N} b_n^3 \int_0^\infty {\rm d}\eta_n^{\rm l} \left(\frac{{\rm d}\hat{\theta}_{n,0}}{{\rm d}\eta_n^{\rm l}} \right)^2 + b_n^4 b_{n-1}^{-1} \int_0^\infty {\rm d}\eta_{n-1}^{\rm l} \tilde{\theta}_n^2(\eta_{n-1}^{\rm l}) + \frac{\alpha_1^4}{\alpha_0 R} \int_0^\infty {\rm d}\eta_0^{\rm u} \tilde{\theta}_1^2(\eta_0^{\rm u}) \quad (33)$$

$$Z_1 = \sum_{n=1}^{N} b_N^{-1} \frac{\alpha_N}{\delta_N^{\rm l}} \frac{\delta_n^{\rm l}}{\alpha_n} \int_0^\infty {\rm d}\eta_n^{\rm l} (1 - {\eta_n^{\rm l}}^2 \hat{\theta}_{n,0} - \tilde{\theta}_{n+1}^2)^2. \quad (34)$$

Varying K_N with respect to the functions $\hat{\theta}_{n,0}$ and $\tilde{\theta}_{n+1}$ we obtain the equations

$$b_n^3 \frac{{\rm d}^2 \hat{\theta}_{n,0}}{{\rm d}{\eta_n^{\rm l}}^2} + K_N b_N^{-1} \frac{\delta_n^{\rm l}}{\alpha_n} \frac{\alpha_N}{\delta_N^{\rm l}} (1 - {\eta_n^{\rm l}}^2 \hat{\theta}_{n,0} - \tilde{\theta}_{n+1}^2) {\eta_n^{\rm l}}^2 = 0, \quad (35)$$

$$b_{n+1}^4 b_n^{-1} \tilde{\theta}_{n+1} - 2 K_N b_N^{-1} \frac{\delta_n^{\rm l}}{\alpha_n} \frac{\alpha_N}{\delta_N^{\rm l}} (1 - {\eta_n^{\rm l}}^2 \hat{\theta}_{n,0} - \tilde{\theta}_{n+1}^2) \tilde{\theta}_{n+1} = 0, \quad (36)$$

analogous to the equations obtained in [5]. Solving (35, 36) we obtain

$$M = K_N^{6/5} (\sigma + \tau) = (1 - b_1^4 - 6\beta^* \sum_{n=1}^{N-1} (b_{n+1}^{10} / b_n)^{1/3}) / b_N^{-1/3} \quad (37)$$

where $\sigma = \int_0^\infty {\rm d}\eta (1 - \eta^2 f)$; $\tau = \int_0^\infty {\rm d}\eta ({\rm d}f/{\rm d}\eta)^2$.

$$\beta^* = (1/6) \int_0^\infty [(1/2)^{5/6} ({\rm d}g/{\rm d}\eta)^2 + (1/2)^{-1/6} (1 - \eta^2 g)] \quad (38)$$

is two times smaller as $\beta = 0.4367$ defined in [5] and $\sigma + \tau = 1.110646$. The coefficients b_n are determined from the equations $\partial M / \partial b_n = 0$ and solving them we obtain

$$b_1 = [(3/4)/(10^N - 1/4)]^{1/4} \quad (39)$$

$$b_{n+1} = 10^{(1/3)[n - (10/9)(1 - 10^{-n})]} \times (2/\beta^*)^{(1/3)(1 - 10^{-n})} b_1^{(1/3)(4 - 10^{-n})}. \quad (40)$$

In order to obtain the F_N as a function of the Rayleigh number we must solve (31) with respect to δ_n^1. As a result we obtain the solution

$$\delta_n^1 = \{R^{-1} \prod_{k=1}^{n-1} [\ln(1/\delta_k^1)]^{6\times 10^{k-1}}\}^{1/(2\times 10^n)} \times \{[2\times 10^n / \ln\{R \prod_{k=1}^{n-1} [\ln(1/\delta_k^1)]^{-6\times 10^{k-1}}\}][1/(1-\Omega_n)]\}^{1/5} \quad (41)$$

with

$$\Omega_n = \{2\times 10^n / \{5 \ln\{R \prod_{k=1}^{n-1} [\ln(1/\delta_k^1)]^{-6\times 10^{k-1}}\}\}\} \times \ln\{2\times 10^n / \{\ln\{R \prod_{k=1}^{n-1} [\ln(1/\delta_k^1)]^{-6\times 10^{k-1}}\}\}\}. \quad (42)$$

Substituting (30–42) into (32) we obtain quite complicated expression for F. In order to evaluate F_N when $N \to \infty$ we shall neglect all of the quantities Ω_n assuming that they are much smaller than 1. We note that some of the functions Ω_n decrease relatively slowly with increasing Rayleigh number. Thus the last assumption affects the boundary layer thicknesses and will result in a higher estimation on the correspondent upper bound on the convective heat transport (for discussion see the next section). For F_N we obtain

$$F_N = [1/(2(\sigma+\tau)]^{6/5}\{10^{(1/3)[N-1-(10/9)(1-1/10^{N-1}]} \times (2/\beta^*)^{(1/3)(1-1/10^{N-1})} b_1^{(1/3)(4-1/10^{N-1})^{2/5}} \times (120 b_1^4 10^{N-2})^{6/5}\}(1/2)^{(2/9)(1-10^{-N})} \times 10^{-(1/405)[10(9N-1)]10^{1-N}} \times R^{(1/3)(1-10^{-N})}(\ln R)^{(2/9)(1-10^{-N})}. \quad (43)$$

If N is very large we can treat $\mu = 10^{-N}$ as a continuous variable. Then we find the optimal N from the requirement $\partial(\ln F_N)/\partial\mu = 0$. Thus $N = (1/\ln(10))[\ln(3/2) + \ln\ln R]$ and the value of F_N when $N \to \infty$ is

$$F \propto [1/(2(\sigma+\tau))]^{6/5}(3/(2\beta^*))^{2/15}(9/10)^{6/5} \times 10^{-38/135}(1/3)^{2/9}R^{1/3} \approx (1/6)R^{1/3}. \quad (44)$$

Figure 2 shows several upper bounds connected to the multi-α-solutions of the variational problem. The application area of the bound obtained by the correspondent multi-α-solution is this one in which the upper bound is larger than the upper bounds obtained by the other multi-α-solutions. Thus the 1-α-solution gives the upper bound on the convective heat transport up to $R = 2.272 \times 10^{12}$ and the region, in which the bound connected to the 2-α-solution of the variational problem is also upper bound on the convective heat transport, is between $R = 2.272\times 10^{12}$ and 7.893×10^{73}. For comparison in the case of a fluid layer with rigid boundaries the bound obtained by the 1-α-solution is upper bound on the heat transport up to $R = 8.945 \times 10^9$ and the region where the bound obtained by the 2-α-solution is upper bound on the convective heat transport is between $R = 8.945\times 10^9$ and $R = 7.866\times 10^{76}$.

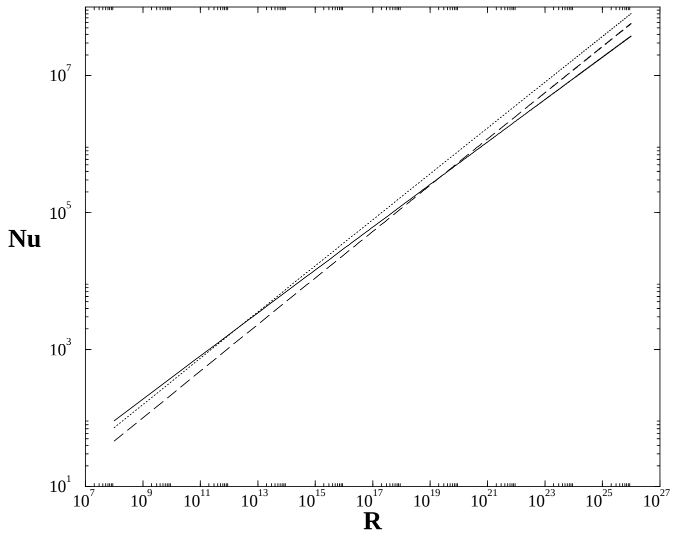

Fig. 2. Nusselt number connected to the multi-α-solutions of the variational problem as a function of the Rayleigh number. Solid line: Nu(R) for the 1-α-solution. Dotted line: Nu(R) for the 2-α-solution. Dashed line: Nu(R) connected to the 3-α-solution of the variational problem.

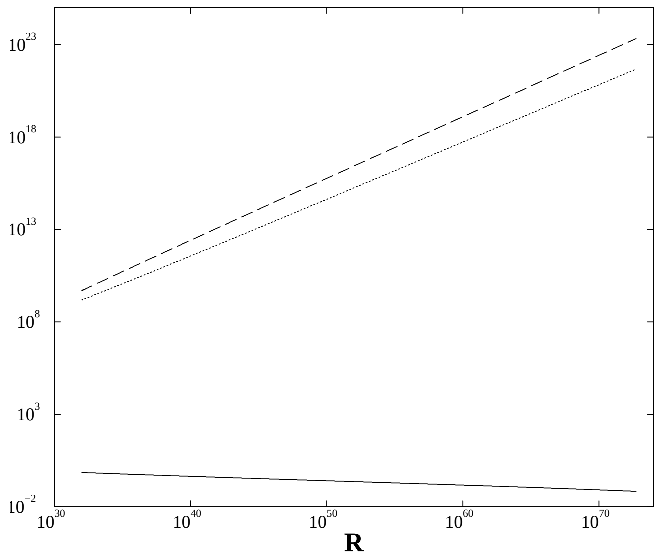

Fig. 3. Several quantities connected to the two-α-solution of the variational problem for the upper bound on the convective heat transport in a fluid layer with two rigid boundaries. Solid line: $\Omega_2(R)$. Dashed line: The upper bound on the Nusselt number obtained by including Ω_2 into the expression for the correspondent boundary layer thickness. Dotted line: The difference between the correspondent bound on the Nusselt number obtained in [5] and the bound obtained in this paper (which is presented by the dashed line in this figure).

4 Discussion

In order to obtain the simplified expression (43) for F_N we have neglected the terms Ω_n in the expressions for the boundary layer thicknesses of the optimum fields connected to the multi-α-solution of the variational problem. This we have obtained the result $\mathrm{Nu}_\infty = 1 + (1/6)R^{1/3}$ for the case of a fluid layer with rigid lower and stress-free upper boundary. Using the same approximation we can obtain the result $\mathrm{Nu}_\infty = 1 + 0.152R^{1/3}$ for the case of a fluid layer with two rigid boundaries. The above two results must be treated as an upper bound on the upper bound on the convective heat transport for the corresponding case. Let us discuss for an example the case of fluid layer with two rigid boundaries. Taking Ω_n into account we obtain for the case of the 2-α-solution of the variational problem

$$\alpha_1 = (R/133)^{1/4};$$
$$\alpha_2 = (2/\beta)^{3/10}(1/133)^{13/40}[\ln(1/\delta_1)]^{1/20}R^{13/40}; \qquad (45)$$
$$\delta_1 = (13D^4/24^4)^{1/20}(1/5)^{-1/5}R^{-1/20}(\ln R)^{-1/5} \qquad (46)$$

with $D = 2.2212$

$$\delta_2 = \{R[\ln(1/\delta_1)]^{-6}\}^{-1/200}\{200/\{\ln\{R[\ln(1/\delta_1)]^{-6}\}\{1/\{1 - (40\ln(200/(\ln(R[\ln(1/\delta_1)]^{-6}))))/(\ln(R[\ln(1/\delta_1)]^{-6}))\}\}\}\} \qquad (47)$$

$$F_2 = (1/133)^{13/100}(2/\beta)^{3/25}[60/(133(\sigma+\tau))]^{6/5} \times [\ln(1/\delta_1)]^{1/50}R^{33/100}/\{200/\{\ln\{R[\ln(1/\delta_1)]^{-6}\}\{1/\{1 - (40\ln(200/(\ln(R[\ln(1/\delta_1)]^{-6}))))/(\ln(R[\ln(1/\delta_1)]^{-6}))\}\}\}\}. \qquad (48)$$

For the case of the three-α-solution we obtain the relationships

$$\alpha_1 = (R/1333)^{1/4};$$
$$\alpha_2 = (2/\beta)^{3/10}(1/1333)^{13/40}[\ln(1/\delta_1)]^{1/20}R^{13/40} \qquad (49)$$
$$\alpha_3 = 10^{3/10}(2/\beta)^{33/100}(1/1333)^{13/40}[\ln(1/\delta_1)]^{1/200} \times [\ln(1/\delta_2)]^{1/20}R^{133/400} \qquad (50)$$

$$\delta_3 = \{R[\ln(1/\delta_1)]^{-6}[\ln(1-\delta_2)]^{-60}\}^{-1/2000} \times \{2000/\{\ln\{R[\ln(1-\delta_1)]^{-6}[\ln(1-\delta_2)]^{-60}\} \times \{1/\{1 - \{400/\{\ln\{R[\ln(1-\delta_1)]^{-6}[\ln(1/\delta_2)]^{-60}\}\} \times \ln\{2000/\ln\{R[\ln(1/\delta_1)]^{-6}[\ln(1/\delta_2)]^{-60}\}\}\}\}\}\}\}^{-1/5} \qquad (51)$$

$$F_3 = \{[\ln(1/\delta_1)]^{1/200}[\ln(1/\delta_2)]^{1/20}R^{133/400}10^{3/25} \times (1/1333)^{133/1000}(2/\beta)^{33/200}[600/(1333(\sigma+\tau))]^{6/5}\}/\delta_3. \qquad (52)$$

Figure 3 shows the effect of the incorporating of Ω_n into the expressions for δ_n. As a result the thicknesses of the boundary layers of the optimum fields increase which leads to lowering of the upper bound on the convective heat transport and to a change in the behaviour of some of the wave numbers α_n of the multi-α-solution with increasing Rayleigh number. In the region of Rayleigh numbers for which the corresponding multi-α-solution of the variational problem gives the upper bound on the convective heat transport this effect is a finite one and moreover we have an accumulation of the corrections of the upper bounds on the heat transport. This is because of the incorporation of the correction, connected to the $(n-1)$-α-solution into the correction connected to the n-α-solution of the variational problem. This incorporation is caused by the simultaneous correction of the thicknesses of the boundary layers of the optimum fields.

Appendix A: The variational functional and the onset of the convection

We can write the variational problem as follows: Given μ find the minimum $R(\mu)$ of the variational functional:

$$\mathcal{R}(w,\theta,\mu) = \frac{\langle|\nabla\theta|^2\rangle}{\langle w\theta\rangle} + \mu\frac{\langle(\overline{w\theta} - \langle w\theta\rangle)^2\rangle}{\langle w\theta\rangle^2} \qquad (A.1)$$

among all fields w, θ that satisfy the condition $\nabla^4 w + \nabla_1^2\theta = 0$ in addition to the rigid lower boundary conditions and stress-free upper boundary conditions.

It can be shown that this functional and the functional (10) are equivalent. We can associate μ with the convective heat transport and R with the Rayleigh number. Thus we can expect that when $\mu = 0$ the functional will contain the solution of the Euler equations and the Rayleigh and wave numbers corresponding to the onset of the convection. Below we show this for the cases of fluid layer with stress-free boundaries, with rigid boundaries and with rigid lower boundary and stress-free upper boundary.

The Euler equations corresponding to the functional (A.1) for the case $\mu = 0$ are

$$\nabla^6\theta + R\nabla^4 w = 0 \qquad (A.2)$$
$$\nabla^4 w + \nabla_1^2\theta = 0. \qquad (A.3)$$

Assuming that the solution is of the kind of 1-α-solution of Busse: $w = w_1(z)\phi_1(x,y)$; $\theta = \theta_1(z)\phi_1(x,y)$ with $\nabla_1\phi_1 = -\alpha_1^2\phi_1$ and $\overline{\phi_1\phi_1} = 1$ we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_1^2\right)^2 w_1 = \alpha_1^2\theta \qquad (A.4)$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_1^2\right)^3 \theta_1 + R\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_1^2\right)^2 w_1 = 0. \qquad (A.5)$$

Introducing the solutions for w_1, θ_1 into above equations we obtain a relationship between the Rayleigh number and the wave number of the 1-α-solution. Through a minimisation we then obtain the values of the Rayleigh number and wave number for the onset of the convection

For the case of a fluid layer with two stress-free boundaries we can perform all calculations analytically. The corresponding solution for w_1 is $w_1 = a \sin[\pi(z+1/2)]$ where a is the amplitude. From (A.4, A.5) we obtain $R = (\alpha^2+\pi^2)^3/\alpha^2$ and the optimisation with respect to α leads us to the onset values $\alpha_{\rm cr} = 2.22144$ and $R_{\rm cr} = 657.51136$. For the rigid-stress-free case the solution for w_1 is [38]:

$$\begin{aligned} w_1 = {} & \sin(7.137877z) \\ & +0.01153032 \sinh(9.110891z)\cos(3.789330z) \\ & +0.00345645 \cosh(9.110819z)\sin(3.789330z) \end{aligned}$$

which leads to the onset values $\alpha_{\rm cr} = 2.6825$ and $R_{\rm cr} = 1100.65$. The analogous calculations for the rigid-rigid case lead to $\alpha_{\rm cr} = 3.117$ and $R_{\rm cr} = 1707.65$.

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